Test 3 Review Key

(1) (a) F, needs to span also.

(b) F, should be $x = P_\beta [x]_\beta$

(c) T, you will get one basis vector for each free variable.

(d) T, row operations do not change the span of the rows.

(e) F, only if the matrix is triangular

(f) F, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2+R_1 \to R_1} \begin{bmatrix} \frac{1}{3} & 0 \\ 2 & 3 \end{bmatrix}$

↑ e.values 1 and 1 ↑ has eigenvalues $\frac{1}{3}$ and 3

(g) F, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable but not invertible.

(h), T, because $u \cdot v = v \cdot u$ (property of dot product)

(2) Relation: $p_1 + p_2 - p_3 = 0$. A basis could be any two of the 3 vectors.

(3) $1-3t+3t^2-t^3$, $4-12t+9t^2$, $3t^2-4t^3$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ -12 \\ 9 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 3 \\ -4 \end{bmatrix}$$

Row reduce $\begin{bmatrix} 1 & 4 & 0 \\ -3 & -12 & 0 \\ 3 & 9 & 3 \\ -1 & 0 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 4 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has a free variable, so they are not lin. indep., and thus do not form a basis.

(4) $\begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 6 & -1 \\ 6 & -2 & -2 \\ -9 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix}}_{=A} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ so the subspace

is equal to Col A. Need a basis for Col A, so

now reduce A to get $A \sim \begin{bmatrix} 3 & 6 & -1 \\ 0 & -14 & 0 \\ 0 & 23 & 0 \\ 0 & 7 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 6 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ first two columns $\begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}$ form a basis for the subspace.

(5) Since H is a subspace we know everything in H is also in V. Since H has a basis of n vectors, it will span an n-dimensional space, which would be all of V.

(6) ${}_{10}\left[\overset{12}{A}\right]$ Maximum rank is 10, so the Null space must have dimension at least 2. If all solutions are multiples of one nonzero solution then dim (Nul A) = 1, which is <u>not possible</u> in this situation.

(7) $\begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & -9 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = 1/3\, x_3$

$x_2 = 1/3\, x_3 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ is a basis.

$x_3 = x_3$

(8) $\det(A - \lambda I) \stackrel{?}{=} \det(A^T - \lambda I) =$

They will be ~~true~~ equal because ~~when you take a determinant~~

~~you can ex~~ $\det(A^T - \lambda I) = \det(A - \lambda I)^T = \det(A - \lambda I)$

$-\lambda I$ only change the diagonal entries.

Property of determinant that $\det A = \det A^T$ (can expand across rows or columns)

(9) $\det\begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix} = 2 - 3\lambda + \lambda^2 - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$

$\Rightarrow D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

- E. space corres. to $\lambda = 5$ is $\begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix}$

  $\Rightarrow$ e. space basis is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- for $\lambda = -2$ get $\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}$

  $x_1 = \frac{-3}{4} x_2$

  $\Rightarrow \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ is basis

$\Rightarrow P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$

(10) (a) $\frac{x \cdot w}{w \cdot w} w = \frac{5}{35}\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3/7 \\ -1/7 \\ 5/7 \end{bmatrix}$

$$\begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = 6 - 6 + 0 = 0 \Rightarrow \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \perp \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

(b) $\left\| \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\| = 10 \Rightarrow \frac{1}{10}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 3/10 \\ 0 \end{bmatrix}$ is unit vector $\perp$ to w.

(11) $y \perp u \Rightarrow y \cdot u = 0$

$y \perp v \Rightarrow y \cdot v = 0$

For w in Span$\{u, v\}$, $w = c_1 u + c_2 v$ for some scalars c_1, c_2.

Then

$$\begin{aligned} y \cdot w &= y \cdot (c_1 u + c_2 v) \\ &= c_1\, y \cdot u + c_2\, y \cdot v \\ &= c_1 (0) + c_2 (0) \\ &= 0 \end{aligned}$$

$\Rightarrow y \perp w.$

So y is orthogonal to any vector w in Span$\{u, v\}$.

Chapt 4 Supp.

#2 Any two of the 3 column vectors will work, same Steps as (4) on the review worksheet

#6 You only need to find two polynomials which are not scaler multiples (this guarantees they are linearly indep).

#8 Did this one in (5) on the review worksheet.

#10 If S was not a basis for V then there would be some vector w not in the Span of S. But then the subset with S and w adjoined would be a larger subset of lin. indep vectors, so S would not be maximal. This is a contradiction.

Chapt. 5 Supp.

#2 $ABx = \lambda x$ for some eigenvalue λ.

Then $BA(Bx) = B(ABx) = B(\lambda x) = \lambda(Bx)$, so

$BA(Bx) = \lambda(Bx) \Rightarrow \lambda$ is an eigenvalue with corresponding eigenvector Bx.

#6 (a) $B = P(5I - 3D + D^2)P^{-1}$